

Noncommutative nonsingular black holes

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Abstract

Adopting noncommutative spacetime coordinates, we determined a new solution of Einstein equations for a static, spherically symmetric matter source. The limitations of the conventional Schwarzschild solution, due to curvature singularities, are overcome. As a result, the line element is endowed of a regular DeSitter core at the origin and of two horizons even in the considered case of electrically neutral, nonrotating matter. Regarding the Hawking evaporation process, the intriguing new feature is that the black hole is allowed to reach only a finite *maximum* temperature, before cooling down to an *absolute zero* extremal state. As a consequence the quantum back reaction is negligible.

1 Introduction

In spite of decades of efforts, a complete and satisfactory quantum theory of Gravity does not yet exist. Thus a great interest has arisen towards the class of model theories able to reproduce quantum gravitational effects, at least in some limit. Quantum Field Theory on a noncommutative manifold or shortly Noncommutative Field Theory (NFT) belongs to such class of model theories. Indeed we retained NFT the low energy limit of String Theory, which is the most promising candidate to be the quantum theory of Gravity.

The starting point of the NFT is the adoption of a noncommutative geometry, namely a manifold whose coordinates may fail to commute in analogy to the conventional noncommutativity among conjugate variables

in quantum mechanics

$$[\mathbf{x}^i, \mathbf{x}^j] = i \theta^{ij} \quad i, j = 1, \dots, n \quad (1)$$

with θ^{ij} an antisymmetric (constant) tensor of dimension (length)². Eq. (1) provides an uncertainty in any measurement of the position of a point on the noncommutative manifold. Indeed we cannot speak of point anymore but rather of delocalized position according to the noncommutative uncertainty. The physical motivation for assuming a noncommutative geometry relies on the bad short distance behavior of field theories, gravitation included. In fact this is a typical feature of theories dealing with point like objects, a problem that has not been completely solved by String Theory too. NFT could provide the solution, since a noncommutative manifold is endowed of a natural cut off due to the position uncertainty. This aspect is in agreement with the long held belief that spacetime must change its nature at distances comparable to the Planck scale. Quantum Gravity has an uncertainty principle which prevents one from measuring positions to better accuracies than the Planck length, indeed the shortest physically meaningful length. In spite of this promising programme and the issue of a seminal paper dated in the early times [1], the interest towards NFT is rather recent. Indeed a significant push forward was given only when, in the context of string theory, it has been shown that target spacetime coordinates become *noncommuting* operators on a D -brane [2]. This feature promoted the interpretation of NFT, among the class of nonlocal field theories [3], as the low energy limit of the theory of open strings.

The inclusion of noncommutativity in field theory in flat space is the subject of a large literature. On the contrary, the purpose of the paper is to introduce noncommutativity effects in the gravitational field, with the hope that noncommutativity could solve the long dated problems of curvature singularity in General Relativity. This investigation is motivated by some mysterious feature of the physics of quantum black holes. Indeed the interest towards their complete understanding have been increasing since the remarkable Hawking discovery about the possibility for them to emit radiation [4]. The general formalism employed is known as quantum field theory in curved space [5]. Such a formalism provides, in terms of a quantum stress tensor $\langle T_{\mu\nu} \rangle$ [6], a satisfactory description of black holes evaporation until the graviton density is small with respect to matter field quanta density. In other words quantum geometrical effects have to be retained negligible, a condition that is not more valid in the terminal stage of the evaporation. The application of noncommutativity to gravity could provide, in an effective way, the still missing description of black holes in those extreme regimes, where stringy effects are considered relevant [7, 8].

2 Noncommutative field theory models

There exist many formulations of NFT, based on different ways of implementing non local deformations in field theories, starting from (1). The most popular approach is founded on the replacement of the point-wise multiplication of fields in the Lagrangian by a non-local Weyl-Wigner-Moyal $*$ -product [9]. In spite of its mathematical exactitude, the $*$ -product NFT suffers non trivial limitations. The Feynman rules, obtained directly from the classical action, lead to unchanged propagators, while the only modifications, concerning vertex contributions, are responsible of the non-unitarity of the theory and of UV/IR mixing. In other words UV divergences are not cured but accompanied by, surprisingly emerging, IR ones. While unitarity can be restored, the restriction of noncommutative corrections only to interaction terms is a non intuitive feature, which appears in alternative formulations [10] too.

Against this background, the coordinate coherent states approach, based on an oscillator representation of noncommutative spacetime, leads to a UV finite, unitary and Lorentz invariant field theory [11]. The starting point of this formulation is to promote the relation (1) to an equation between Lorentz tensors

$$[\mathbf{x}^\mu, \mathbf{x}^\nu] = i \theta^{\mu\nu} \quad \mu, \nu = 0, \dots, n \quad (2)$$

Thus $\theta^{\mu\nu}$, now an antisymmetric Lorentz tensor, can be represented in terms of a block diagonal form

$$\theta^{\mu\nu} = \text{diag} \left(\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_{n/2} \right) \quad \mu, \nu = 0, \dots, n \quad (3)$$

where $\hat{\theta}_i = \theta_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In case of odd dimensional manifold the last term on the diagonal is zero. In other words, for the covariance of (2) we provide a foliation of spacetime into noncommutative planes, defined by (3). There is a condition to be satisfied: field theory Lorentz invariance and unitarity implies that noncommutativity does not privilege any of such planes, namely there is a unique noncommutative parameter $\theta_1 = \theta_2 = \dots \theta_{n/2} = \theta$.

The other key ingredient of this approach is the interpretation of conventional coordinates as mean values of coordinate operators subject to (1), to take into account the quantum geometrical fluctuations of the spacetime manifold. Mean values are calculated over coherent states, which results eigenstates of ladder operator, built with noncommutative coordinates only. In the absence of common eigenstates, the choice of coherent states is motivated by the fact that they result the states of minimal uncertainty and

provide the best resolution of the position over a noncommutative manifold. In other words, the effective outcome is the loose of the concept of point in favor of smeared position on the manifold. At actual level, the delocalization of fields is realized by deforming the source term of their equations of motion, namely substituting Dirac delta distribution (local source) with Gaussian distribution (nonlocal source) of width $\sqrt{\theta}$. As a result, the ultraviolet behavior of classical and quantum fields [12] is cured.

3 The noncommutative black hole

To provide a black hole description by means of a noncommutative manifold, one should know how to deal with the corresponding gravity field equations. Fortunately, this nontrivial problem can be circumvented by a noncommutative deformation of only the matter source term, leaving unchanged the Einstein tensor. This procedure, already followed in $(1+1)$ dimensions [13] and $(3+1)$ linearized General Relativity [14], is in agreement with the general prescription to obtain nonlocal field theories from noncommutativity [11]. This line of reasoning is supported by the following motivations. Noncommutativity is an intrinsic property of a manifold and affects matter and energy distribution, by smearing point-like objects, also in the absence of curvature. On the other hand, the metric is a geometrical device defined over the underlying manifold, while curvature measure the metric intensity, as a response to the presence of mass and energy distribution. Being the energy-momentum tensor the tool which gives the information about the mass and energy distribution, we conclude that, in General Relativity, noncommutativity can be taken into account by keeping the standard form of the Einstein tensor in the l.h.s. of the field equations and introducing a modified source term in the r.h.s..

Therefore we assume, as mass density of the noncommutative delocalized particle, the Gaussian function of minimal width $\sqrt{\theta}$

$$\rho_{\theta}(\vec{x}) = \frac{M}{(4\pi\theta)^{3/2}} \exp\left(-\vec{x}^2/4\theta\right) \quad (4)$$

Thus the particle mass M is *diffused* throughout a region of linear size $\sqrt{\theta}$ taking into account the intrinsic uncertainty encoded in the coordinate commutator (2). The distribution function $\rho_{\theta}(\vec{x})$ is static, spherically symmetric and exponentially vanishing at distances $r \gg \sqrt{\theta}$. In this limit $\rho_{\theta}(\vec{x})$ reproduces point-like sources and leads to the conventional Schwarzschild solution. On these grounds, we are looking for a noncommutative version of a with $T_0^0 = \rho_{\theta}(\vec{x})$ as source of Einstein equations. There are two further conditions to be taken: the covariant conservation

of the energy-momentum tensor $\nabla_\nu T^{\mu\nu} = 0$ and $g_{00} = -g_{rr}^{-1}$ to preserve a Schwarzschild-like property. Therefore the solution of Einstein equations is¹:

$$ds^2 = \left(1 - \frac{4M}{r\sqrt{\pi}}\gamma\right) dt^2 - \left(1 - \frac{4M}{r\sqrt{\pi}}\gamma\right)^{-1} dr^2 - r^2 d\Omega^2 \quad (5)$$

where $\gamma \equiv \gamma(3/2, r^2/4\theta) = \int_0^{r^2/4\theta} dt t^{1/2} e^{-t}$ is the lower incomplete Gamma function. This line element describes the geometry of a noncommutative black hole and should give us useful insights about possible noncommutative effects on Hawking radiation. Let's start our analysis from the presence of

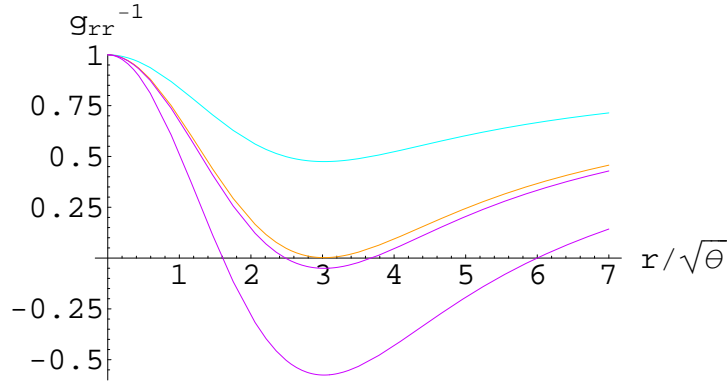


Figure 1: g_{rr}^{-1} vs r , for various values of $M/\sqrt{\theta}$. Intercepts on the horizontal axis give radii of the event horizons. $M = \sqrt{\theta}$, (cyan curve) no horizon; $M = 1.9\sqrt{\theta}$, (yellow curve) one *degenerate* horizon $r_0 \approx 3.0\sqrt{\theta}$, *extremal* black hole; $M = 3\sqrt{\theta}$ (magenta curve) two horizons.

eventual event horizons. Since in our case, the equation $g_{00}(r_H) = 0$ cannot be solved in closed form, one can numerically determine their radius by plotting g_{00} . Figure (1) shows that noncommutativity introduces new behavior with respect to standard Schwarzschild black hole. Instead of a single event horizon, there are different possibilities: (a) two distinct horizons for $M > M_0$ (yellow curve); (b) one degenerate horizon in $r_0 = 3.0 \times \sqrt{\theta}$, with $M = M_0 = 1.9 \times (\sqrt{\theta})/G$ corresponding to *extremal black hole* (cyan curve); (c) no horizon for $M < M_0$ (violet curve). In view of these results, there can be no black hole if the original mass is less than the *minimal mass* M_0 . Furthermore, contrary to the usual case, there can be *two horizons* for large

¹We use convenient units $G_N = 1$, $c = 1$.

masses. By increasing M , i.e. for $M \gg M_0$, the *inner horizon* shrinks to zero, while the *outer* one approaches the Schwarzschild value $r_H = 2M$.

For what concerns the covariant conservation of $T^{\mu\nu}$, one finds that such requirement leads to

$$T^\theta_\theta \equiv \partial_r (r T^r_r) = -\rho_\theta(r) - r \partial_r \rho_\theta(r). \quad (6)$$

The emerging picture of is that of a *self-gravitating, droplet* of *anisotropic fluid* of density ρ_θ , radial pressure $p_r = -\rho_\theta$ and *tangential pressure* $p_\perp = -\rho_\theta - r \partial_r \rho_\theta(r)$. We are not dealing with a massive, structure-less point. Thus results reasonable that a non-vanishing radial pressure balances the inward gravitational pull, preventing droplet to collapse into a matter point. This is the basic physical effect on matter caused by spacetime noncommutativity and the origin of all new physics at distance scale of order $\sqrt{\theta}$. Regarding the physical interpretation of the pressure, we underline that it does not correspond to the inward pressure of outer layers of matter on the core of a “star”, but to a totally different quantity of “quantum” nature. It is the outward push, which is conventionally defined to be negative, induced by noncommuting coordinate quantum fluctuations. In a simplified picture, such a quantum pressure is the relative of the cosmological constant in DeSitter universe. As a consistency check of this interpretation we are going to show that line element (5) is well described near the origin by a DeSitter geometry.

Let us now consider the black hole temperature $T_H \equiv \left(\frac{1}{4\pi} \frac{dg_{00}}{dr} \right)_{r=r_H} :$

$$T_H = \frac{1}{4\pi r_H} \left[1 - \frac{r_H^3}{4\theta^{3/2}} \frac{e^{-r_H^2/4\theta}}{\gamma(3/2; r_H^2/4\theta)} \right] \quad (7)$$

For large black holes, i.e. $r_H^2/4\theta \gg 1$, one recovers the standard result for the Hawking temperature $T_H = \frac{1}{4\pi r_H}$. At the initial state of radiation the black hole temperature increases while the horizon radius is decreasing. It is crucial to investigate what happens as $r_H \rightarrow \sqrt{\theta}$. In the standard (commutative) case T_H diverges and this puts limit on the validity of the conventional description of Hawking radiation. Against this scenario, temperature (7) includes noncommutative effects which are relevant at distances comparable to $\sqrt{\theta}$. Behavior of the temperature T_H as a function of the horizon radius is plotted in Fig.(2). In the region $r_H \simeq \sqrt{\theta}$, T_H deviates from the standard hyperbola. Instead of exploding with shrinking r_H , T_H reaches a maximum in $r_H \simeq 4.7\sqrt{\theta}$ corresponding to a mass $M \approx 2.4\sqrt{\theta}/G_N$, then quickly drops to zero for $r_H = r_0 = 3.0\sqrt{\theta}$ corresponding to the radius of the extremal black hole in figure (1). In the region $r < r_0$ there is no black hole and the corresponding temperature cannot be

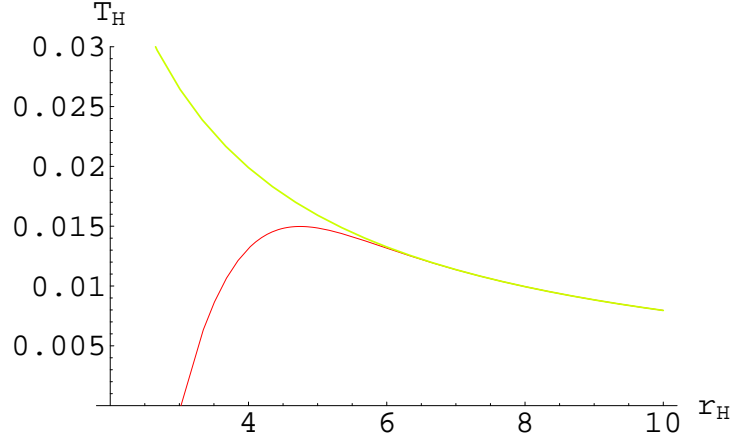


Figure 2: Plot of T_H vs r_H , in $\sqrt{\theta}$ units. yellow curve is the plot of (7): $T_H = 0$ for $r_H = r_0 = 3.0\sqrt{\theta}$, i.e. for the extremal black hole, while the maximum temperature $T_H \simeq 0.015 \times 1/\sqrt{\theta}$ corresponds to a mass $M \simeq 2.4 \times \sqrt{\theta}$. For comparison, we plotted in red the standard Hawking temperature. The two temperatures coincide for $r_H > 6\sqrt{\theta}$.

defined. As a summary of the results, the emerging picture of non commutative black hole is that for $M \gg M_0$ the temperature is given by the Hawking temperature (3) with negligibly small exponential corrections, and increases, as the mass is radiated away. T_H reaches a maximum value at $M = 2.4\sqrt{\theta}$ and then drops as M approaches M_0 . When $M = M_0$, $T_H = 0$, event horizon is degenerate, and we are left with a “frozen” extremal black hole.

At this point, important issue of Hawking radiation back-reaction should be discussed. In commutative case one expects relevant back-reaction effects during the terminal stage of evaporation because of huge increase of temperature [6, 15]. As it has been shown, the role of noncommutativity is to cool down the black hole in the final stage. As a consequence, there is a suppression of quantum back-reaction since the black hole emits less and less energy. Eventually, back-reaction may be important during the maximum temperature phase. In order to estimate its importance in this region, let us look at the thermal energy $E = T_H \simeq 0.015/\sqrt{\theta}$ and the total mass $M \simeq 2.4\sqrt{\theta} M_{Pl}^2$. In order to have significant back-reaction effect T_H^{Max} should be of the same order of magnitude as M . This condition leads to the estimate $\sqrt{\theta} \approx 0.2 l_{Pl} \sim 10^{-34} cm$. Expected values of $\sqrt{\theta}$ are well above the Planck length l_{Pl} , while the back-reaction effects are suppressed even if $\sqrt{\theta} \approx 10 l_{Pl}$ and $T_H^{Max} \approx 10^{16} GeV$. For this reason we can safely

use unmodified form of the metric (5) during all the evaporation process.

Finally, we would like to clarify what happens if the starting object has mass smaller than M_0 , with particular attention to the eventual presence of a *naked singularity*. To this purpose we are going to study the curvature scalar near $r = 0$. The short distance behavior of R is given by

$$R(0) = \frac{4M}{\sqrt{\pi}\theta^{3/2}} \quad (8)$$

For $r \ll \sqrt{\theta}$ the curvature is actually *constant and positive*. Thus, an eventual naked singularity is replaced by a DeSitter, *regular* geometry around the origin. Earlier attempts to avoid curvature singularity at the origin of Schwarzschild metric have been made by matching DeSitter and Schwarzschild geometries both along time-like [16], and space-like matter shells [17], or constructing regular black hole geometries by-hand [18]. In our approach, it is noncommutativity that induces a smooth and continuous transition between the two geometries.

4 Concluding remarks

The above results show that the coordinate coherent state approach to noncommutative effects can cure the singularity problems at the terminal stage of black hole evaporation.

In particular we have shown that noncommutativity, being an intrinsic property of the manifold itself, can be introduced in General Relativity by modifying the matter source. The Energy-momentum required for this description is of form of the ideal fluid, although a non-trivial pressure is invoked. In spite of complicated equation of state it can be studied in the regions of interest and new black hole behavior is discovered in the region $r \simeq \sqrt{\theta}$. Specifically, we have shown that there is a minimal mass $M_0 = 1.9\sqrt{\theta}$ to which a black hole can decay through Hawking radiation. The reason why it does not end up into a naked singularity is due to the finiteness of the curvature at the origin. The everywhere regular geometry and the residual mass M_0 are both manifestations of the Gaussian delocalization of the source in the noncommutative spacetime. On the thermodynamic side, the same kind of regularization takes place eliminating the divergent behavior of Hawking temperature. As a consequence there is a maximum temperature that the black hole can reach before cooling down to absolute zero. As already anticipated in the introduction, noncommutativity regularizes divergent quantities in the final stage of black hole evaporation in the same way it cured UV infinities in noncommutative quantum field theory. We have also estimated that back-reaction does not modify the original metric in a significant manner.

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